

(2+1)-Dimensional Black Hole with Coulomb-like Field.

Mauricio Cataldo^{a *}, Norman Cruz^{b †}, Sergio del Campo^{c ‡} and Alberto García^{d §}

^a Departamento de Física, Facultad de Ciencias, Universidad del Bío-Bío, Avda. Collao 1202, Casilla 5-C, Concepción, Chile

^b Departamento de Física, Facultad de Ciencia, Universidad de Santiago de Chile, Avda. Ecuador 3493, Casilla 307, Santiago, Chile.

^c Instituto de Física, Facultad de Ciencias Básicas y Matemáticas, Universidad Católica de Valparaíso, Avda. Brasil 2950, Valparaíso, Chile

^d Departamento de Física, CINVESTAV-IPN, Apartado Postal 14-740, C.P. 07000, México, D.F. México

Abstract: A (2+1)-static black hole solution with a non-linear electric field is derived. The source to the Einstein equations is a nonlinear electrodynamics, satisfying the weak energy conditions, and it is such that the energy momentum tensor is traceless. The obtained solution is singular at the origin of coordinates. The derived electric field $E(r)$ is given by $E(r) = q/r^2$, thus it has the Coulomb form of a point charge in the Minkowski spacetime. This solution describes charged (anti)-de Sitter spaces. An interesting asymptotically flat solution arises for $\Lambda = 0$.

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Most of the papers on 2+1-solutions coupled to electromagnetic fields are done for Maxwell electrodynamics [1,4], i.e. the electromagnetic tensor is derived, in 2+1 theory, from a Lagrangian which is proportional to the single invariant, namely $L \propto F$, with $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. On the other hand, the introduction of electrodynamics of nonlinear type [5–7], where the dependence on the invariant is enlarged, has proved to be fruitful [8].

In 3+1 electromagnetism, the Maxwell energy momentum tensor is given by $T_{\mu\nu} = (F_{\mu\alpha}F_{\nu}^{\alpha} - g_{\mu\nu}F)/(4\pi)$ and consequently is trace free. If one constructs a nonlinear electrodynamics based on the invariant F , i.e. $L = L(F)$, the resulting energy momentum tensor is given as $T_{\mu\nu} = g_{\mu\nu}L_F - F_{\mu\alpha}F_{\nu}^{\alpha}L$ and its trace becomes $T = 4L_F - 4FL$. Thus, if one demands that the trace to be vanished, i.e. $T = 0$, we find that $L_F - FL = 0$, whose solution is $L \propto F$. In Minkowski spacetime the Maxwell theory is singled out among all nonlinear theories by the vanishing of the trace. Recall that in 3+1 there is a second invariant – a pseudoscalar $\check{G} = \frac{1}{4}\check{F}_{\mu\nu}F^{\mu\nu}$, where \star stands for the duality operation. We may include into the Lagrangian for building up a wider nonlinear theories in which the Born-Infeld electrodynamics [9] is an example.

In 2+1 spacetime, the Maxwell energy momentum tensor is of the same form as in 3+1 dimensions, but the

trace contrary of the 3+1 case occurs to be non-vanishing, i.e. $T = F/4\pi \neq 0$. Hence, this 2+1 Maxwell theory has always trace. The electric field for a static circularly symmetric metric coupled to a Maxwell field occurs to be proportional to the inverse of r , i.e. $E(r) \propto 1/r$, hence the potential A_0 is logarithmic, i.e. $A_0 \propto \ln r$, and consequently blows up at $r = 0$ and r going to infinity.

In this paper we are interested in electromagnetic theories in which the energy momentum tensor is traceless. This condition restricts the class of nonlinear electrodynamics to be studied. Incidentally the traceless nonlinear electrodynamics, in 2+1 dimensions, occurs to be unique with Lagrangian proportional to $F^{3/4}$. Restricting our study to a static circularly symmetric metric, the resulting electromagnetic field for this theory surprisingly is proportional to the inverse of r^2 , i.e. a Coulomb law for a point charge in 3+1 Minkowski space. Moreover, the energy momentum tensor fulfills the weak energy conditions. The resulting metric depends on three parameters: the mass, M , the cosmological constant, Λ , and the charge, q . When $\Lambda < 0$, for certain range of values of these constants, it describes black holes, i.e. charged anti-de Sitter spacetimes with inner and outer horizons. These horizons are roots of an algebraic cubic equation. In this family, when the horizons shrink to a single one, r_{extr} , one obtains an extreme black hole. If these roots are complex we obtain a naked singular solution. For $\Lambda > 0$ there is only one positive root and the corresponding gravitational field describes a solution with a cosmological horizon. A detailed analysis depending on the values of the constants is given. The behavior of the g^{rr} is plotted for different branches. It is worthwhile to point out that this metric allows for asymptotically flat solution for vanishing Λ . Finally, thermodynamics aspects for the studied metric are explicitly worked out.

We are using electromagnetic Lagrangian depending upon a single invariant $L(F)$, where $F = 1/4F^{ab}F_{ab}$. To deal with physically reasonable theories, the fulfillment of the weak energy conditions is imposed on the corresponding energy momentum tensor: for any timelike vector u^a , $u^a u_a = -1$ (we are using signature $-++$) one requires $T_{ab}u^a u^b \geq 0$ and $q_a q^a \leq 0$, where $q^a = T_b^a u^b$. The action of the (2+1)-Einstein theory coupled with nonlinear electrodynamics is given by

*E-mail address: mcataldo@alihuén.ciencias.ubiobio.cl

†E-mail address: ncruz@lauca.usach.cl

‡E-mail address: sdelcamp@ucv.cl

§E-mail address: alberto.garcia@fis.cinvestav.mx

$$S = \int \sqrt{-g} \left(\frac{1}{16\pi} (R - 2\Lambda) + L(F) \right) d^3x, \quad (1)$$

with arbitrary, at this stage, the electromagnetic Lagrangian $L(F)$. We are using units in which $c = G = 1$. Since there is an ambiguity in the definition of the gravitational constant (there is not Newtonian gravitational limit in (2+1)-dimensions) one can maintain the factor $1/16\pi$ in the action to keep the parallelism with (3+1)-gravity. The variation with respect to the metric gives the Einstein equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}, \quad (2)$$

$$T_{ab} = g_{ab}L(F) - F_{ac}F_b{}^c L_{,F}, \quad (3)$$

while the variation with respect to the electromagnetic potential A_a entering in $F_{ab} = A_{b,a} - A_{a,b}$, yields the electromagnetic field equations

$$\nabla_a (F^{ab} L_{,F}) = 0, \quad (4)$$

where $L_{,F}$ denotes the derivative of $L(F)$ with respect to F . In what follows we shall restrict ourselves to the study of the nonlinear field such that the energy momentum tensor (3) has vanishing trace. The trace of this tensor (3) occurs to be

$$T = T_{ab}g^{ab} = 3L(F) - 4FL_{,F}, \quad (5)$$

thus, if one requires T to vanish, one obtains

$$L = C |F|^{3/4}, \quad (6)$$

where C is a constant of integration, and bars denote absolute value. One can rewrite this Lagrangian as

$$L = C \left| \frac{1}{2} (B^2 - E^2) \right|^{3/4}, \quad (7)$$

when referred to orthonormal local lorentzian basis.

In what follows, we shall look for black hole solutions in the static case. The metric we are dealing with is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (8)$$

where $f(r)$ is an unknown function of the variable r . Next, we assume that the electromagnetic field is given by: the electric field $E_r = F_{tr}$ and the magnetic scalar field $B := F_{\phi r}$. The corresponding Maxwell equations are

$$\frac{d}{dr} [rEL_{,F}] = 0, \quad \frac{d}{dr} \left[\frac{f}{r} BL_{,F} \right] = 0. \quad (9)$$

By virtue of the Einstein equations the B field has to vanish. In fact, the Ricci tensor components, evaluated for the metric (8), yield the following relation

$$A := R_{tt} + f^2 R_{rr} = 0, \quad (10)$$

while the evaluation of the same relation using the electromagnetic energy-momentum gives

$$A = -8\pi L_{,F} \left(\frac{f}{r} B \right)^2. \quad (11)$$

Therefore, the scalar magnetic field B has to be equated to zero, $B = 0$. Hence, for the metric (8), the only case allowed is just the one with the electric field E . Consequently, the electromagnetic field tensor can be given as

$$F_{ab} = E(r) (\delta_a^t \delta_b^r - \delta_a^r \delta_b^t). \quad (12)$$

The invariant F occurs to be

$$2F = -E^2(r). \quad (13)$$

Integrating the equation (9) for E , one obtains

$$E(r) L_{,F} = -\frac{q}{4\pi r}. \quad (14)$$

where q is an integration constant. Using now (13) we express the derivative $L_{,F}$ as function of r , as follows

$$L_{,r} = \frac{q}{4\pi r} E_{,r}. \quad (15)$$

From (7), setting $B = 0$, we arrive at $L = C E^{3/2}$. Entering this L into (15) one gets $E = (q^2/6\pi C)^{2/3} 1/r^2$. Choosing now $C = \sqrt{|q|}/6\pi$, the electric field becomes

$$E(r) = \frac{q}{r^2}, \quad (16)$$

which coincides with the standard Coulomb field for a point charge of the Maxwell theory in the four dimensional Minkowski space. The Lagrangian in this case is given by

$$L = \frac{q^2}{6\pi r^3} = \frac{\sqrt{|q|}}{6\pi} E^{3/2}. \quad (17)$$

It is easy to check up that the energy momentum tensor for this Lagrangian satisfies the weak energy conditions: $q_a q^a \leq 0$, where $q^a = T_b^a u^b$ for any timelike vector u^a , which in terms of the related quantities is equivalent to the inequality

$$-(L + E^2 L_{,F}) = \frac{q^2}{12\pi r^3} \geq 0. \quad (18)$$

Having established the electric field, we are now ready to search for solutions of the Einstein equations, which equivalently are

$$R_{ab} = 8\pi T_{ab} + 2\Lambda g_{ab}, \quad (19)$$

where it has been considered that the trace of (3) is equal to zero, i.e, $T = 0$. These equations, for $R_{tt} (= -f^2 R_{rr})$ and $R_{\Omega\Omega}$ components, yield respectively the equations:

$$f_{,rr} + \frac{f_{,r}}{r} = -2\Lambda + \frac{2q^2}{3r^3}, \quad (20)$$

$$f_{,r} = -2\Lambda r - \frac{4q^2}{3r^2}. \quad (21)$$

It is easy to show that equation (20), by virtue of the equation (21) is just an identity. Therefore, the only Einstein equation to be integrated is (21), which gives

$$f(r) = D - \Lambda r^2 + \frac{4q^2}{3r}, \quad (22)$$

where D is a constant of integration. We will see now that the constant D can be expressed in terms of the mass at infinity. To demonstrate this, we shall use the quasilocal formalism developed by Brown et al. [10,11] to evaluate the quasilocal energy and mass of a stationary and axisymmetric asymptotically non-flat spacetime. For the circularly symmetric metric (8) the quasilocal energy $\mathcal{E}(r)$ and the quasilocal mass $M(r)$ at a radial boundary r can be shown to be respectively

$$\mathcal{E} = 2(\sqrt{f_0(r)} - \sqrt{f(r)}), \quad (23)$$

$$M(r) = \mathcal{E}(r)\sqrt{f(r)}, \quad (24)$$

where $f_0(r) = g_0^{rr}(r)$ is a background metric function which determines the zero of the energy. The function $f_0(r)$ can be obtained simply by assigning some special values to the constants of integration, determining this way the reference space-time. We set in (22) $q = D = 0$, arriving at the background space, which corresponds to an asymptotic anti-de Sitter spacetime. The same background function was used in [11,12] for analogous calculations. For $\Lambda = -1/l^2 < 0$, we get $\sqrt{f_0(r)} = r/l$ and then the quasilocal energy and the quasilocal mass are given respectively by

$$\mathcal{E}(r) = \frac{r}{l} - \sqrt{D + \frac{r^2}{l^2} + \frac{4}{3} \frac{q^2}{r}}, \quad (25)$$

$$M(r) = 2\frac{r}{l} \sqrt{D + \frac{r^2}{l^2} + \frac{4}{3} \frac{q^2}{r}} - 2 \left(D + \frac{r^2}{l^2} + \frac{4}{3} \frac{q^2}{r} \right). \quad (26)$$

As $r \rightarrow \infty$, the analogous ADM mass is defined to be $M =: M(\infty)$. In our case we see from (25) and (26) that $\mathcal{E}(\infty)$ vanishes and $M(\infty) =: M = -D$ correspondingly. The constant D can be thought of as the asymptotic observable mass, $M > 0$, and we can write

$$f(r) = -M + \frac{r^2}{l^2} + \frac{4q^2}{3r}. \quad (27)$$

The case $\Lambda > 0$ corresponds to an asymptotically de-Sitter spacetime.

For vanishing cosmological constant, i.e. $\Lambda = 0$, one has an asymptotically flat solution coupled with a Coulomb-like field.

To establish the existence of horizons, one has to require the vanishing of the g^{rr} component, i.e., $f(r) = -M - \Lambda r^2 + \frac{4q^2}{3r} = 0$, for handling all possible cases. The roots of this equation are

$$r_1 = \frac{h}{3\Lambda} - \frac{M}{h}, \quad (28)$$

$$r_2 = -\frac{h}{6\Lambda} + \frac{M}{2h} + \frac{i\sqrt{3}}{2} \left(\frac{h}{3\Lambda} + \frac{M}{h} \right), \quad (29)$$

$$r_3 = -\frac{h}{6\Lambda} + \frac{M}{2h} - \frac{i\sqrt{3}}{2} \left(\frac{h}{3\Lambda} + \frac{M}{h} \right), \quad (30)$$

where

$$h = \left[\left(18q^2 + 3\sqrt{3 \left(\frac{M^3 + 12q^4\Lambda}{\Lambda} \right)} \right) \Lambda^2 \right]^{1/3}. \quad (31)$$

These equations give the location of the horizons (if there are any). Since the coordinate r range from 0 to infinity, we exclude the negative roots. As it is well known from the properties of cubic equations, the complex or real character of the roots depends crucially on the sign of the radical $\alpha := M^3/\Lambda + 12q^4$. Due to $M > 0$, the complex or real character of these roots depends on values of Λ . For $\alpha > 0$, or $\Lambda > -M^3/12q^4$, we get one real and two complex roots. A more detailed study shows that the real root is r_1 and positive. The remaining roots, r_2 and r_3 , are complex.

Notice that $f(r)$ decreases for $r > 0$, as we can see from FIG. (1). In this plot we have taken the value $M = 1$ (we have also included negative values of M for showing the different behavior of the roots). Notice also that we have a cosmological horizon.

For $\alpha < 0$ (or $\Lambda < -M^3/12q^4$) there are three real roots. On the other hand, if $\Lambda < 0$ the corresponding roots r_2 and r_3 can be complex or real. More exactly, for $-M^3/12q^4 < \Lambda < 0$ we see that the roots r_2 and r_3 are complex. For $\Lambda < -M^3/12q^4$ we have real roots only. In this case we may write r_2 and r_3 as

$$r_{\pm} = -2\sqrt{\frac{-M}{3\Lambda}} \cos \left(\frac{1}{3} \arccos \left[\frac{2q^2}{\sqrt{-\frac{M^3}{3\Lambda}}} \right] \pm \frac{2\pi}{3} \right). \quad (32)$$

These roots represent the horizons of a black hole. The outer horizon (the event horizon) r_+ and the inner horizon r_- . If $\alpha = 0$, or $\Lambda = -M^3/12q^4$, then it is obtained an extreme black hole. In such case the mass becomes $M = M_{extr} = -(12q^4\Lambda)^{1/3}$ and the roots are

$$r_1 = 2 \left(\frac{2q^2}{3\Lambda} \right)^{1/3}, \quad r_2 = r_3 = r_{extr} = - \left(\frac{2q^2}{3\Lambda} \right)^{1/3}. \quad (33)$$

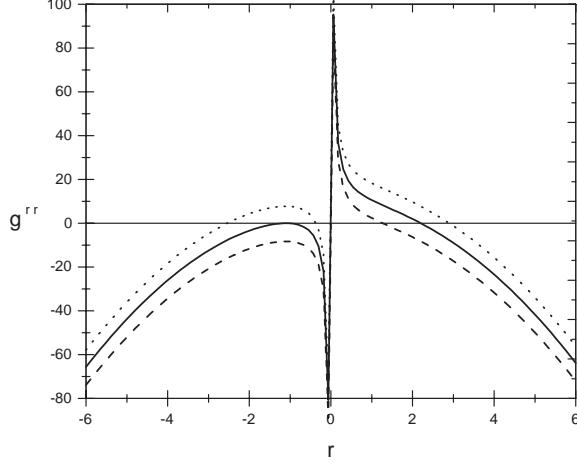


FIG. 1. Behavior of g^{rr} for different values of M and $\Lambda > 0$. The continuous line shows the behavior of the extreme case, $M_{extr} = -7.268$, $\Lambda = 2$ and $q = 2$. For the same values of Λ and q , the dashed and the dotted lines show the behavior for $M = -15$ and $M = 1$, respectively.

From these expressions we see that for $\Lambda > 0$ there is cosmological horizon. In FIG. (2) we show different situations for negative $\Lambda < 0$. Firstly, the solid line represent an extreme black hole (where we have used the values $M = 7.268$, $\Lambda = -2$ and $q = 2$). Secondly, the dashed line represents a naked singularity (with values $M = 1$, $\Lambda = -2$ and $q = 2$). Finally, the dotted line correspond to a black hole with two horizons (here, $M = 25.15$, $\Lambda = -2$ and $q = 2$). When $q = 0$ we get $r_- = 0$ and $r_+ = \sqrt{-M/\Lambda}$, which are the horizons of the anti de Sitter 2+1 metric.

An interesting case results when $\Lambda = 0$. Here, we have a charged solution with a cosmological horizon at

$$r_H = \frac{4q^2}{3M}. \quad (34)$$

Notice that the gravitational field is asymptotically flat in this case. As far as we know, there are no other solutions with cosmological horizon which exhibits the same property.

In order to study the thermodynamics of our solutions, we evaluate the temperature of the black hole, which is given in terms of its surface gravity [11,13]

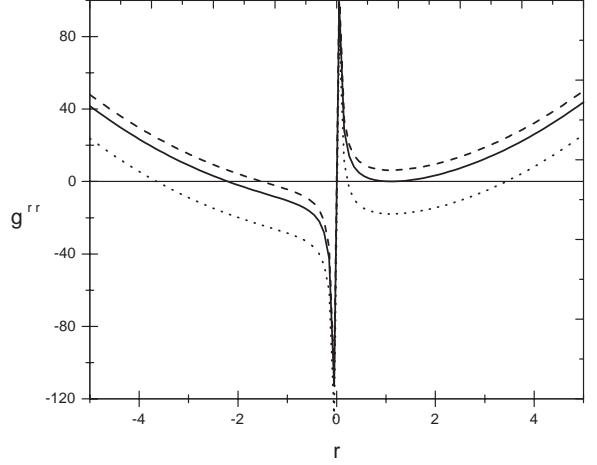


FIG. 2. Behavior of g^{rr} for different values of M and for $\Lambda < 0$. The continuous line shows the behavior of the extreme case, $M_{extr} = 7.268$, $\Lambda = -2$ and $q = 2$. For the same values of Λ and q , the dashed and the dotted lines show the behavior for $M = 1$ and $M = 25.15$, respectively.

$$k_B T_H = \frac{\hbar}{2\pi} k. \quad (35)$$

For a circularly symmetric metric the surface gravity can be computed via [14,15]

$$k = \frac{1}{2} \left| \lim_{r \rightarrow r_+} \frac{\partial_r g_{tt}}{\sqrt{-g_{tt} g_{rr}}} \right|, \quad (36)$$

where r_+ is the event horizon. From our solution and $\Lambda = -1/l^2 < 0$ we get that

$$k_B T = \frac{\hbar}{4\pi} \times \left| -\sqrt{\frac{16M}{3l^2}} \cos \left(\frac{1}{3} \arccos \left[\frac{2q^2}{\sqrt{\frac{M^3 l^2}{3}}} \right] + \frac{2\pi}{3} \right) - \frac{q^2}{M l^2} \cos^{-2} \left(\frac{1}{3} \arccos \left[\frac{2q^2}{\sqrt{\frac{M^3 l^2}{3}}} \right] + \frac{2\pi}{3} \right) \right|. \quad (37)$$

It is easy to check that when $q = 0$, T reduces to the static BTZ temperature [16]. In the extreme case (33), the temperature vanishes.

If $\Lambda = 0$ the temperature at the cosmological horizon becomes $T = \frac{3\hbar}{16\pi} \frac{M^2}{q^2}$.

The entropy of the black hole can be trivially obtained by using $S = 4\pi r_+$. Other thermodynamic quantities such as heat capacity and chemical potential can be computed as well [11]. We do not discuss them here.

Our solution is singular only at $r = 0$. Effectively, the first two invariant curvature scalars are

$$R = 6\Lambda \quad (38)$$

$$R_{ab}R^{ab} = 12\Lambda^2 + \frac{8q^4}{3r^6}. \quad (39)$$

As we mentioned above, there is a true singularity at the origin of the coordinate system. Notice that these invariants are nonsingular at the horizons.

In this paper we have investigated nonlinear electrodynamics restricted to traceless energy momentum in 2+1 dimensions. The obtained solutions for a circularly symmetric metric describe black holes with a Coulomb-like field, which are asymptotically anti-de Sitter spacetimes. It is worthwhile to point out that the derived charged black holes possess finite mass contrary to the charged BTZ solutions, which because of the presence of the logarithmic term in the metric yields a divergent quasilocal mass. The solution with $\Lambda = 0$ is remarkable since it has a cosmological horizon and the spacetime is asymptotically flat.

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